IDEMPOTENTS IN QUASIGROUPS OF BOL–MOUFANG TYPE

ALEKSANDAR KRAPEŽ
MATHEMATICAL INSTITUTE OF SANU, BELGRADE SERBIA

Quasigroups are algebras $(S; \cdot, \backslash, /)$ satisfying identities:

$$x \backslash xy = y \qquad\qquad xy/y = x$$
$$x(x\backslash y) = y \qquad\qquad (x/y)y = x$$

If a quasigroup has a *left (right) neutral element* e so that:

$$(\mathrm{e}\cdot) \qquad ex = x \qquad\qquad (\cdot\mathrm{e}) \qquad xe = x$$

then it is called *left (right) loop*. If the neutral element is two–sided, i.e. both (e·) and (·e) are true, we have *loops*. If there is an $e \in S$ such that

$$\text{(i)} \qquad ee = e$$

we say that e is an idempotent and that $(S; \cdot, \backslash, /, e)$ is a *quasigroup with an idempotent*. If all elements of S are idempotents, i.e.

$$\text{(I)} \qquad xx = x$$

we have an *idempotent quasigroup*. We are also interested in quasigroups which have the property of *commutativity*:

$$\text{(c)} \qquad xy = yx.$$

Identities (i), (e·), (·e), (I), (c) (or their equivalents) are *basic*. Quasigroup variety defined by a subset of basic identities is a *basic variety*. There are ten basic varieties.

An identity $s = t$ is of *Bol–Moufang type* if:

(1) the only operation in s, t is $\cdot$,
(2) the same three variables appear in both s, t and in the same order,
(3) one of the variables appear twice in both s and t,
(4) the remaining two variables appear once in s and once in t.

There are 90 identities of Bol–Moufang type, 30 of them trivial (i.e. equivalent to $x = x$). They are defined (for loops) and partially classified in the papers [1] and [2] by F. Fenyves. They characterize important varieties such as Moufang, left (right) Bol, extra loops, groups and many others.

Varieties defined by a combination of basic identities and a single identity of Bol–Moufang type are called the varieties of Bol–Moufang type. The classification of loops of Bol–Moufang type was settled in the paper [4] by J. D. Phillips and P. Vojtěchovský. It was found that there are 14 varieties of loops of Bol–Moufang type (plus the variety of all loops). Phillips and Vojtěchovský also classified varieties of quasigroups of Bol–Moufang type. Beside the variety of all quasigroups, there are 26 such varieties. In the same paper we can find results of the classification of the varieties of commutative quasigroups (loops) of Bol–Moufang type. There are 7 (6) such varieties, making the total of 44 varieties.

We investigate the six families of quasigroups of Bol–Moufang type generated by the basic varieties which were not considered by Phillips and Vojtěchovský. It turned out that there are only 19 new varieties.

Various relationships between the varieties of Bol–Moufang type are verified by the theorem prover `Otter` and the model builder program `Mace` both written by W. W. McCune [3].

References

[1] F. Fenyves: Extra loops I, Publ. Math. Debrecen 15, (1968), 235–238.

[2] F. Fenyves: Extra loops II. On loops with identities of Bol–Moufang type, Publ. Math. Debrecen 16, (1969), 187–192.

[3] W. W. McCune: Programs `Otter` and `Mace2`, Argonne National Laboratory, (2003), http://www-unix.mcs.anl.gov/AR/

[4] J. D. Phillips, P. Vojtěchovský: The varieties of loops of Bol–Moufang type, submitted to Alg. Univ.

[5] J. D. Phillips, P. Vojtěchovský: The varieties of quasigroups of Bol–Moufang type: An equational reasoning approach, submitted to J. of Algebra